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TECHNICAL REPORT

WSRL-0178-TR

THE EQUIVALENCE OF LINEARLY CONSTRAINED OPTIMUM ARRAY WEIGHT VECTORS UNDER DIFFERENT OPTIMISATION CRITERIA

D.A. GRAY

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Three optimisation criteria are considered: minimisation of total output power, least mean square fit to a signal vector, and maximisation of signal-to-noise ratio. For practical purposes it is often necessary to apply some form of constraint to the weighting vectors. The optimum narrowband array weight vector is derived for each of the three criteria when an arbitrary set of linear constraints are imposed on the weight vector.

The equality of the derived weight vectors in most cases of practical interest is proved.

Finally, some examples are considered which show how a number of recently proposed weighting vectors can be derived as special cases of the constrained solutions.

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The performance of arrays of receivers can be improved by optimisation of the weighting vectors used to form the output beams.

Three optimisation criteria are considered: minimisation of total output power, least mean square fit to a signal vector, and maximisation of signal-to-noise ratio. For practical purposes it is often necessary to apply some form of constraint to the weighting vectors. The optimum narrowband array weight vector is derived for each of the three criteria when an arbitrary set of linear constraints are imposed on the weight vector.

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1. INTRODUCTION

The choice of array shading weights modifies the performance of a delay and sum beamformer. In conventional beamforming the choice of a set of weighting vectors involves a tradeoff between the width of the main beam and the height of unwanted sidelobes.

If the noise field is known 'a priori' the performance can be optimised in some sense by a correct choice of the weighting vectors.

This choice of an optimum set of array shading weights has received considerable attention(ref.1,2,3) and the weight vectors are often chosen to satisfy optimisation criteria such as:

(1) minimising the total output power, or

(2) a least mean square fit to the signal vector, or

(3) maximisation of the signal-to-noise ratio

In general a single point constraint of a fixed (non zero) response to signals from the steered direction is incorporated and under this constraint the three optimisation criteria result in the same weight vector(ref.1). This constraint ensures that signal output power increases as the input signal-to-noise ratio increases. A further extension has been to derive optimum weight vectors which satisfy an arbitrary number of linear constraints(ref.2,3). An example of this is the use of derivative constraints to maintain the main lobe response(ref.4) and hence prevent cancellation of desired signals close to the steered direction. The optimisation criterion used in these cases has been that of minimising the total output power(ref.2,3) and maximising the signal-to-noise ratio(ref.4).

In this paper the optimum weight vectors for multiple linear constraints are derived for the three different optimisation criteria discussed above. Under some very mild restrictions on the choices of constraints, it is shown that all three criteria result in the same weight vector for a given set of constraints. The frequency domain formulation is used to show this but the results can easily be extended into the time domain.

Some special examples are studied to show how the general solutions reduce to a number of processors that have appeared in the literature(ref.5,6,7). In particular when the problem is overconstrained (ie the number of constraints is greater than the number of receivers) then the optimum weight vectors reduce to some quadratic estimators that have recently been derived in the literature (ref.6,7). In these cases the optimum weighting vectors are independent of the noise crosspower spectral matrix and are determined by the array geometry.

This work is part of a continuing R&D programme in signal processing for underwater acoustics and has been carried out under tast DST 79/069.

2. DERIVATION

Let $x = \{x_j\}$ be the vector of outputs of the K receivers at some frequency f and be $w = \{w_j\}$ be the vector of receiver weights. The covariance matrix R (K x K) is defined* by

---

* The superscript ^{H} denotes the Hermition transpose of a matrix or vector.

$$R = \langle xx^H \rangle .$$

In the absence of a signal the receiver noise outputs are denoted by n and the noise covariance matrix is defined by

$$R_n = \langle nn^H \rangle$$

and

$$R = R_S + R_n .$$

The following derivations have been formulated for the frequency domain but can easily be extended to the time domain by a suitable redefinition of the matrices and vectors.

Any set of L linear constraints on the weight vector w can be expressed as

$$Mw = c \tag{1}$$

where M is the L x K constraint matrix and c is the L x 1 vector of constraint values.

2.1 Minimum power

This criterion chooses the weights, w_i, such that the total output power ie

$$w^H Rw,$$

is minimised subject to the constraints (1) being satisfied.

Introducting the vector λ of undefined Lagrangian multipliers the cost function H(w) to be minimised is given by

$$H(w) = w^H Rw - (Mw - c)^H \lambda - \lambda^H (Mw - c).$$

Differentiating the above equation with respect to w^H and equating the result to zero*, the following equation is obtained:

$$Rw - M^H \lambda = 0.$$

Assuming R^{-1} exists then the optimum weight vector, denoted as $w_o^{(MP)}$, is given by

$$w_o^{(MP)} = R^{-1} M^H \lambda \tag{2}$$

*It can readily be proved that differentiating with respect to w^H and w is equivalent to differentiating with respect to the real and imaginery components separately.

Furthermore requiring that the constraint equation (1) is satisfied implies that

$$(MR^{-1}M^H)\lambda = c \tag{3}$$

If M is of rank L (ie of full rank) then it follows that

$$\lambda = (MR^{-1}M^H)^{-1}c$$

and hence

$$w_o^{(MP)} = R^{-1}M^H (MR^{-1}M^H)^{-1}c$$

which has been derived by a similar argument in reference 2. However, more generally it follows that any solution to equation (3) can be written in the form

$$\lambda = (MR^{-1}M^H)^+c + (I - (MR^{-1}M^H)^+ (MR^{-1}M^H))z$$

where z is an arbitrary vector, and A^+ denotes the Moore-Penrose pseudoinverse of A (see reference 8 or Appendix I for definition of A^+).

In this case equation (2) becomes

$$w_o^{(MP)} = R^{-1}M^H (MR^{-1}M^H)^+c + R^{-1}M^H (I - (MR^{-1}M^H)^+ (MR^{-1}M^H))z$$

which reduces to

$$w_o^{(MP)} = R^{-1}M^H (MR^{-1}M^H)^+c \tag{4}$$

since (see Appendix I) the second term is equal to zero. The output power is given by

$$\begin{aligned} w_o^{(MP)^H} R\, w_o^{(MP)} &= c^H (MR^{-1}M^H)^+ MR^{-1} RR^{-1}M^H (MR^{-1}M^H)^+c \\ &= c^H(MR^{-1}M^H)^+ MR^{-1}M^H (MR^{-1}M^H)^+c \\ &= c^H(MR^{-1}M^H)^+c \end{aligned}$$

by virtue of equations (I.1) (ie the definition of the Moore-Penrose pseudoinverse). Note that both the expression for the optimum weighting vector and the minimum output power can be derived directly from p.49 of reference 9.

2.2 Least mean square derivation

The error between some desired output, d, and the weighted receiver outputs is given by

$$\epsilon = d - w^H x .$$

The mean square value of this error can be shown to be given by

$$\begin{aligned} \langle \epsilon^2 \rangle &= d^H d - w^H \langle d\, x \rangle - \langle x^H d \rangle w + w^H R w \\ &= d^H d - w^H v - v^H w + w^H R w \end{aligned} \quad (5)$$

where the covariance between d, the response due to a signal, and x, the receiver outputs is simply the steering vector, v, if the signal and noise are uncorrelated. Thus minimising equation (5) subject to the constraints implies a cost function

$$H(w) = d^H d + w^H R w - v^H w - w^H v + (c^H - M^H w^H)\lambda + \lambda^H (c - Mw)$$

Differentiating with respect to w^H and equating to zero implies that $w_o^{(LMS)}$ is given by

$$w_o^{(LMS)} = R^{-1} (M^H \lambda + v) \quad (6)$$

Ensuring that $w_o^{(LMS)}$ satisfies the constraints implies that

$$(MR^{-1}M^H)\lambda = c - MR^{-1} v$$

and hence λ is given by

$$\lambda = (MR^{-1}M^H)^+ (c - MR^{-1}v) + (I - (MR^{-1}M^H)^+ (MR^{-1}M^H))z$$

where z is arbitrary.

Substituting for λ in equation (6) it can be shown in a manner similar to that used in Section 2.1 that

$$w_o^{(LMS)} = R^{-1}M^H (MR^{-1}M^H)^+ c + R^{-1} v - R^{-1}M^H (MR^{-1}M^H)^+ MR^{-1} v$$

It can also be shown that the output power is given by

$$w_o^{(LMS)} R w_o^{(LMS)} = c^H (MR^{-1}M^H)^+ c + v^H R^{-1} v - v^H R^{-1}M^H (MR^{-1}M^H)^+ MR^{-1} v$$

2.3 Maximisation of beam signal to noise ratio

If the signal covariance matrix is defined by R_s then the beam SNR is given by

$$g_w = \frac{w^H R_s w}{w^H R_n w}$$

Maximising the SNR subject to constraint (1) implies the cost function

$$H(w) = \frac{w^H R_s w}{w^H R_n w} + (c^H - w^H M^H)\lambda + \lambda^H(c - Mw)$$

Assuming R_n to be non-singular, differentiating H(w) with respect to w^H and equating the result to zero implies that

$$P\, w_o^{(SNR)} = -f(w_o)\, M^H\lambda$$

where

$$P = R_n - R_s/g_o$$

and

$$f(w) = (w^H R_n w)^2 / w^H R_s w$$

and

$$g_o = \frac{w_o^{H(SNR)} R_s w_o^{(SNR)}}{w_o^{H(SNR)} R_n w_o^{(SNR)}}$$

By an analogous derivation to the previous sections it follows that

$$w_o^{(SNR)} = P^{-1}M^H (MP^{-1}M^H)^{-1} c \qquad (7)$$

provided both P and $MP^{-1}M^H$ are non-singular.

Now since R_s is given by vv^H then it follows that

$$P = R_n - \frac{vv^H}{g_o}$$

and is non-singular unless $g_o = v^H R_n^{-1} v$. (From p.29 of reference 9

$$| P | = | R_n | (g_o - v^H R_n^{-1} v)).$$

This special case corresponds to the single constraint of unity response in the look direction and in this case w_o can be derived by the method in reference 1, or by an application of the theorem on p.48 of reference 9.

Unfortunately equation (7) expresses $w_o^{(SNR)}$ as a function of g_o which in turn is defined by $w_o^{(SNR)}$. In Appendix II this explicit dependance is removed and $w_o^{(SNR)}$ is shown to be given by

$$w_o^{(SNR)} = R_n^{-1} M^H (MR_n^{-1} M^H)^{-1} c + \frac{c^H (MR_n^{-1} M^H)^{-1} c}{c^H y} (R_n^{-1} v - R_n^{-1} M^H (MR_n^{-1} M^H)^{-1} MR_n v) \tag{8}$$

where

$$y = (MR_n^{-1} M^H)^{-1} MR_n^{-1} v$$

As is also shown in Appendix II the optimum gain g_o is given by

$$g_o = g_v - y^H MR_n^{-1} v + \frac{c^H y \, y^H c}{c^H (MR_n^{-1} M^H)^{-1} c}$$

where g_v is defined by

$$g_v = v^H R_n^{-1} v$$

and, as discussed earlier (and in reference 1), g_v is the output SNR of an optimum processor subject only to the single constraint of unity response in the look direction. Furthermore if unity response in the look direction is one of the L multiple constraints then it can readily be shown that

$$g_o = \frac{1}{c^H (MR^{-1} M^H)^{-1} c}$$

3. RELATIONSHIPS BETWEEN THE PROCESSORS

All optimum weight vectors derived are of the form

$$S^{-1}M^H(MS^{-1}M^H)^{-1}c + \alpha(S^{-1}v - S^{-1}M^H(MS^{-1}M^H)^{-1} MS^{-1}v)$$

where

(i) $S = R$ and $\alpha = 0$ for minimum power criterion,

(ii) $S = R$ and $\alpha = 1$ for least mean squares criterion, and

(iii) $S = R_n$ and $\alpha = \dfrac{c^H(MR_n^{-1} M^H)^{-1}c}{c^H y}$ for maximum signal to noise ratio criterion.

Now if it can be shown that

$$S^{-1}v - S^{-1}M^H(MS^{-1}M^H)^{-1} MS^{-1}v = 0 \tag{9}$$

then it follows that all weight vectors are of the form

$$S^{-1} M^H (MS^{-1} M^H)^{-1} c.$$

Expression (9) can be shown to be equal to zero provided one row of the constraint matrix M is equal to (or a multiple of) v^H.

This can be simply * proved by noting that M can always be rearranged such that v^H is in the first row. Thus by taking the first column of the equation

$$(MS^{-1} M^H)^{-1} (MS^{-1} M^H) = I,$$

it follows that $$(MS^{-1} M^H)^{-1} (MS^{-1}v) = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

*The author is indebted to Dr A.K. Steele who simplified an original proof of this.

Thus

$$S^{-1} M^H (MS^{-1} M^H)^{-1} (MS^{-1} v) = S^{-1} M^H \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

$$= S^{-1} v.$$

Since v determines the look direction this amounts to fixing the response to a known plane wave signal.

Thus provided the same constraint matrix M is used to estimate $w_o^{(MP)}$ and $w_o^{(LMS)}$ *and one row of this matrix is given by* v^H then it follows that

$$w_o^{(MP)} = w_o^{(LMS)} .$$

Furthermore on substituting

$$R = R_n + vv^H$$

in the equation

$$w_o^{(MP)} = R^{-1} M^H (MR^{-1} M^H)^{-1} c$$

and using Woodbury's Identity for $(R_n + vv^H)^{-1}$ it follows that

$$w_o^{(MP)} = R_n^{-1} M^H (MR_n^{-1} M^H)^{-1} c + a(R_n^{-1} M^H (MR_n^{-1} M^H)^{-1} MR_n^{-1} v - R_n^{-1} v)$$

where a is a constant, and so

$$w_o^{(MP)} = R_n^{-1} M^H (MR_n^{-1} M^H)^{-1} c$$

$$= w_o^{(SNR)} .$$

Thus provided the same constraints are imposed and a fixed look direction constraint is imposed it follows that all these estimators are identical.

4. EXAMPLES

In this section some examples of the processors subject to some obvious physical constraints are given.

4.1 Unity (or fixed) response in the look direction

It can readily be shown that for a processor of the form $R^{-1}v$ the mean beam power output decreases as the signal strength increases. This can be avoided by imposing the constraint of a fixed response in the look direction, ie

$$v^H w_o = 1$$

Thus

$$M = v^H$$

and

$$c = 1$$

Then

$$w_o^{(MP)} = R^{-1}v(v^H R^{-1}v)^{-1}$$

$$= w_o^{(LMS)},$$

since

$$R^{-1}v - R^{-1} M^H(MR^{-1}M^H)^{-1}MR^{-1}v$$

$$= R^{-1}v - R^{-1}v$$

$$= 0.$$

Note that P as defined in Section 2.3 is singular and hence

$$w_o^{(SNR)} = P^{-1}M^H (MP^{-1}M^H)^{-1}c$$

is ill-defined. However from reference 1* it follows that

*Note however that direct substitution in equation (8) gives the correct solution in this case.

$$w_o^{(SNR)} = \frac{R^{-1}v}{(v^H R^{-1} v)}.$$

4.2 Fixed response in the look direction and a null steered in a specified direction

A particular processor that has received some attention(ref.5) maximises the signal to noise ratio in a direction v while steering a fixed null in another direction specified by u. If, as in the previous section a fixed (say unity) response is imposed in the look direction then the constraint matrix and vector are given by

$$M = \begin{pmatrix} v^H \\ u^H \end{pmatrix}$$

and

$$c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

respectively.

From equation (4) it follows that

$$w_o^{(MP)} = R^{-1}\ (v,u) \left(\begin{pmatrix} v^H \\ u^H \end{pmatrix} R^{-1}\ (v,u) \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (R^{-1}v, R^{-1}u) \begin{pmatrix} g_v & g_{uv} \\ g_{vu} & g_u \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where

$$g_v = v^H R^{-1} v,\ g_u = u^H R^{-1} u,$$

and

$$g_{uv} = g_{vu}^{*} = v^H R^{-1} u.$$

Thus

$$w_o^{(MP)} = \frac{(R^{-1}v, R^{-1}u)}{(g_u g_v - g_{uv} g_{vu})} \begin{pmatrix} g_u & -g_{uv} \\ -g_{vu} & g_v \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{g_u}{g_u g_v - g_{uv} g_{vu}} \left(R^{-1} v - \frac{g_{vu} R^{-1} u}{g_u} \right)$$

which apart from the scaling factor is the null steering weight vector derived by Fenwick(ref.5).

Now

$$c^H (MR^{-1}M^H)^{-1} c = \frac{(1,0)}{g_u g_v - g_{uv} g_{vu}} \begin{pmatrix} g_u & -g_{uv} \\ -g_{vu} & g_v \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{g_u}{g_u g_v - g_{uv} g_{vu}} \quad .$$

Thus since

$$g_o = \frac{1}{c^H (MR^{-1}M^H)^{-1} c}$$

it follows that

$$g_o = g_v - \frac{g_{uv} g_{vu}}{g_u}$$

which again agrees with the expression derived in reference 5.

Also it can readily be verified that

$$R^{-1}v - R^{-1}M^H (MR^{-1}M^H)^{-1} MR^{-1} v = 0$$

thus demonstrating the general result proven in Section 3 that

$$w_o^{(MP)} = w_o^{(LMS)} = w_o^{(SNR)} \quad .$$

4.3 Fixed response in look direction and K-1 independent nulls

In this problem all the degrees of freedom are incorporated in the K x K non-singular constraint matrix, M, where the columns of M are the phase vectors corresponding to the K directions.

It then follows that

$$w_o^{(MP)} = R^{-1}M^H (MR^{-1}M^H)^{-1} c$$

$$= R^{-1}M^H (M^H)^{-1} RM^{-1} c$$

$$= M^{-1} c \tag{10}$$

Now in references 6 and 7 some generalised linear processors have been derived on the assumption that the incident distribution consists of N plane waves. These estimators were derived on the basis of statistical considerations and deconvolution of the array response respectively. Defining the K x N matrix whose columns are the phase vectors corresponding to the N assumed directions as V, then a general form(ref.6) for the weighting vector in the i^{th} direction is

$$w_i = V^{+H} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \\ \\ \end{array} \tag{11}$$

In particular when N = K and the directions are independent then

$$w_i = V^{-1H} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \end{array}$$

which as can readily be seen is a special case of equation (10). The fact that

$$c^H = (0, \ldots, 1, \ldots, 0)$$

readily demonstrates the observation of reference 6 that for a given steer direction this processor steers K-1 nulls in the other assumed plane wave arrival directions.

4.4 Overconstrained case

The matrix M in this case is L x K and L>K. To obtain consistent solutions the rank of M must be less than or equal to K. If M is chosen to be of full rank then

$$M^+ = (M^H M)^{-1} M^H$$

and hence

$$M^{+}M = (M^{+}M)^{H} = I$$

The L x L matrix $MR^{-1}M^{H}$ is singular and so the Moore-Penrose pseudoinverse must be used. Thus

$$w_o^{(MP)} = R^{-1}M^{H}(MR^{-1}M^{H})^{+}c$$

However from Appendix I it follows that

$$(MR^{-1}M^{H})^{+} = M^{+H}RM^{+}$$

Thus

$$w_o^{(MP)} = R^{-1}M^{H}M^{+H}RM^{+}c$$

$$= M^{+}c$$

$$= (M^{H}M)^{-1}Mc$$

As in the previous section the choice of M=V and $c^{H} = (0, \ldots, 1, \ldots, 0)$ reduces this estimator to the Least Mean Square estimator derived in reference 6 and to a special case of the estimator derived in reference 7 by an iterative deconvolution of the array response.

5. SUMMARY

The problem of constrained optimisation for three different criteria ie Minimum Power, Least Mean Squares and Maximum Signal to Noise ratio has been solved under some very general conditions.

In all cases of practical interest and provided identical constraints are imposed, the three processors have been shown to be identical.

A number of examples which demonstrate this equivalence have also been given. Furthermore these examples show how a number of well known processors are all special cases of the constrained optimum processors. In particular if the weight vectors are critically or over constrained then weight vectors reduce to the ideal deconvolving weights which have been derived recently.

This paper, in common with previous work has assumed that the covariance matrix is non-singular. This is not always necessary and further work is proceeding to derive similar expressions which do not require non-singularity of the covariance matrix.

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APPENDIX I

APPLICATION OF THE MOORE-PENROSE PSEUDOINVERSE

I.1 Let G be any L x K matrix then the Moore-Penrose pseudoinverse G^+ is defined by the following equations:

$$GG^+G = G$$

$$G^+GG^+ = G^+$$

and

$$(GG^+)^H = GG^+$$

$$(G^+G)^H = G^+G$$

Define the vector u by

$$u = R^{-1} M^H (I - (MR^{-1}M^H)^+ (MR^{-1}M^H))z$$

where z is any arbitrary vector and R is a K x K positive definite matrix and M is any L x K matrix. Denoting $(MR^{-1}M^H)$ by A it follows that

$$u^H Ru = z^H (I-A^+A)^H MR^{-1}RR^{-1} M^H (I - A^+A)z$$

$$= z^H (I-A^+A)^H A(I-A^+A) z$$

$$= 0 .$$

Thus since R is positive definite it follows that

$$u = 0$$

(Note this also follows directly from a lemma on p.22 of reference 10).

I.2 If M is of full column (ie L > K) rank it follows that

$$M^+M = M^HM^{+H} = I$$

and also defining

$$\tilde{A} = M^{+H} RM^+$$

it follows that

$$\tilde{A}A = M^{+H} M^{H} = MM^{+} \qquad (I.2)$$

and

$$A\tilde{A} = MM^{+}$$

Thus

$$(\tilde{A}A)^{H} = \tilde{A}A$$

and

$$(A\tilde{A})^{H} = A\tilde{A}$$

Also

$$\tilde{A}A\tilde{A} = M^{+H} RM^{+}MM^{+}$$

$$= \tilde{A}$$

and similarly

$$A\tilde{A}A = A \ .$$

Hence $\tilde{A}$ is A^{+} the Moore Penrose pseudoinverse of A where $A = MR^{-1}M^{H}$. (In this case it can readily be proved by direct substitution of equation (I.2) that

$$M^{H} (I - A^{+}A) z = 0$$

and hence $u = 0$).

APPENDIX II

CONSTRAINED MAXIMUM SNR PROCESSOR

Since P is defined by

$$P = R_n - R_s/g_o$$

where

$$g_o = \frac{w_o^H R_s w_o}{w_o^H R_n w_o}$$

it follows that

$$w_o^H P w_o = 0 .$$

Since w_o is given by

$$w_o = P^{-1} M^H (M P^{-1} M^H)^{-1} c$$

this then implies that

$$c^H (M P^{-1} M^H)^{-1} c = 0. \qquad (II.1)$$

If R_s is equal to vv^H then by Woodbury's identity it follows that

$$P^{-1} = R_n^{-1} + \frac{R_n^{-1} vv^H R_n^{-1}}{a}$$

where

$$a = g_o - g_v \qquad (II.2)$$

and

$$g_v = v^H R_n^{-1} v.$$

Furthermore it holds that

$$(MP^{-1}M^{H})^{-1} = (MR_n^{-1}M^{H})^{-1} - \frac{yy^H}{\beta}$$

where

$$y = (MR_n^{-1}M^{H})^{-1} MR_n^{-1}v$$

and

$$\beta = v^H R_n^{-1}M^H (MR_n^{-1}M^{H})^{-1} MR_n^{-1}v + a.$$

Thus equation (II.1) becomes

$$c^H(MR_n^{-1}M^{H})^{-1}c = \frac{c^H yy^H c}{(v^H R_n^{-1}M^H (MR_n^{-1}M^{H})^{-1} MR_n^{-1}v + a)}$$

Thus

$$a = \frac{c^H yy^H c}{c^H(MR_n^{-1}M^{H})^{-1}c} - v^H R_n^{-1} M^H(MR_n^{-1}M^{H})^{-1} MR_n^{-1}v.$$

Rearranging equation (II.2) and substituting the above expression for a gives

$$g_o = g_v - v^H R_n^{-1}M^H (MR_n^{-1}M^{H})^{-1} MR_n^{-1}v + \frac{c^H yy^H c}{c^H(MR_n^{-1}M^{H})^{-1}c} \qquad \text{(II.3)}$$

Hence

$$w_o = P^{-1}M^H(MP^{-1}M^{H})^{-1}c$$

$$= \left\{R_n^{-1} + \frac{R_n^{-1}vv^H R_n^{-1}}{a}\right\}M^H\left\{(MR_n^{-1}M^{H})^{-1} - \frac{yy^H}{\beta}\right\}c$$

where β can be shown to be given by

$$\beta = \frac{c^H y y^H c}{c^H (M R_n^{-1} M^H)^{-1} c} = y^H (M R_n^{-1} M^H) y + a$$

Thus

$$w_o = R_n^{-1} M^H (M R_n^{-1} M^H)^{-1} c - \frac{y^H c}{\beta} R_n^{-1} M^H y - \frac{y^H c}{\beta\, a} (y^H (M R_n^{-1} M^H) y - \beta) R_n^{-1} v$$

$$= R_n^{-1} M^H (M R_n^{-1} M^H)^{-1} c + \frac{y^H c}{\beta} (R_n^{-1} v - R_n^{-1} M^H (M R_n^{-1} M^H)^{-1} M R_n^{-1} v)$$

$$= R_n^{-1} M^H (M R_n^{-1} M^H)^{-1} c + \frac{c^H (M R_n^{-1} M^H)^{-1} c}{c^H y} (R_n^{-1} v - R_n^{-1} M^H (M R_n^{-1} M^H)^{-1} M R_n^{-1} v) .$$

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